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Conditionally Exactly Solvable Potentials: A Supersymmetric Construction Method

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We present in this paper a rather general method for the construction of so-called conditionally exactly solvable potentials. This method is based on algebraic tools known from supersymmetric quantum mechanics. Various families of one-dimensional potentials are constructed whose corresponding Schrödinger eigenvalue problem can be solved exactly under certain conditions of the potential parameters. Examples of quantum systems on the real line and the half line as well as on some finite interval are studied in detail. © 1998 Academic Press

1. INTRODUCTION

Since the advent of quantum mechanics there has been interest in quantum models whose corresponding Schrödinger equation can be solved exactly. To be more precise, by exactly solvable we mean that the spectral properties, that is, the eigenvalues and eigenfunctions, of the Hamiltonian characterizing the quantum system under consideration can be given in an explicit and closed form. The most important examples are the harmonic oscillator and the hydrogen atom. An first attempt in finding such systems has been initiated by Schrödinger [1] himself and is now know as the factorization method [2]. This factorization method has been revived during the last two decades in connection with supersymmetric quantum mechanics [3]. In particular, the factorization condition which is a condition on the quantum mechanical potential for its exact solvability has been rediscovered and is now known as the so-called shape-invariance condition [4]. In fact, there have been several attempts in finding additional shape-invariant potentials besides those already given by Infeld and Hull [2].

In the classical papers by Natanzon [5] and Ginocchio [6] it has been shown that one can go even far beyond the class of shape-invariant potentials. To be more

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explicit, the Natanzon class of potentials is the most general one for which the Schrödinger eigenvalue problem can be reduced to a hypergeometric differential equation. The members of the Ginocchio class of potentials, which is a subclass of the previous one, are of particular interest in nuclear physics [6]. The relation between these classes of potentials and supersymmetry has been studied in detail by Cooper *et al.* [7].

Other methods which are also closely related to supersymmetric (SUSY) quantum mechanics are based on the idea of finding pairs of (essentially) isospectral Hamiltonians [8–12]. One of these methods, the Darboux method, is based on the existence of an operator A and its adjoint $A^\dagger$ which act as transformation operators between a pair of self-adjoint Hamiltonians $H_\pm$ [13, 14]:

$$AH_- = H_+ A, \qquad H_- A^\dagger = A^\dagger H_+. \tag{1.1}$$

Obviously, H_+ and H_- are essentially isospectral, that is, there spectra coincide except for a possible additional vanishing eigenvalue. Knowing, for example the eigenfunctions of H_+ one can immediately obtain those of H_- with the help of the transformation operator $A^\dagger$. This Darboux method, which has originally been applied with linear first-order differential operators A, has recently been extended to higher-order differential operators where it is called N-extended Darboux transformation (with N standing for the highest order of the momentum operator appearing in A) [15, 16].

Another different method for constructing exactly solvable systems has been suggested by Abraham and Moses [8] and is based on the inverse method. As in the Darboux method one starts with a given exactly solvable Hamiltonian and constructs a new one whose spectral properties follow from those of the starting Hamiltonian. Applying this approach to SUSY quantum systems it is equivalent to the Darboux method [10].

In this paper we develop yet another method for constructing so-called conditionally exactly solvable systems [17]. This method, which is based on the SUSY formulation of one-dimensional quantum systems has recently been suggested by us in [18]. It is the aim of this paper to present the detailed ideas of this approach and to apply it to various physically relevant model systems on the real line, the half line, and those on a finite interval. In particular, we will show that many of the newly found exactly solvable potentials contain as special cases also those found by the other two methods mentioned above.

In the next section we will briefly review the basic algebraic tools of SUSY quantum mechanics [3], which will be used in the general construction method presented in Section 3. The remaining three sections present a detailed discussion of examples on the real line, the half line and finite intervals. To be more explicit, in Section 4 we construct the most general class (within our approach) of SUSY partner potentials for the linear harmonic, the Morse and the symmetric Rosen–Morse oscillator. Section 5 contains the corresponding results for the radial harmonic oscillator and the radial Coulomb problem. In Section 6 we consider the symmetric Pöschl–Teller oscillator as an example on the finite interval $[-\pi/2, \pi/2]$.

2. SUPERSYMMETRIC QUANTUM MECHANICS

In this section we briefly review the basic concepts of Witten's model of supersymmetric quantum mechanics [19, 3]. This model consists of a pair of standard Schrödinger Hamiltonians

$$H_\pm = -\frac{1}{2}\frac{d^2}{dx^2} + V_\pm(x) \tag{2.1}$$

which act on the Hilbert space $\mathscr{H}$ of square integrable functions over a given configuration space. In this paper we will consider systems on the real line $\mathbb{R}$, on the positive half line $\mathbb{R}^+$, and on the finite interval $x \in [-\pi/2, \pi/2]$. In the latter two cases we will impose Dirichlet boundary conditions, that is, the Hilbert spaces are given by $\mathscr{H} = L^2(\mathbb{R})$, $\mathscr{H} = \{\psi \in L^2(\mathbb{R}^+) \mid \psi(0) = 0\}$, and $\mathscr{H} = \{\psi \in L^2([-\pi/2, \pi/2]) \mid \psi(\pm\pi/2) = 0\}$, respectively. The so-called SUSY partner potentials in (2.1) are expressed in terms of the real-valued SUSY potential W and its derivative $W' = dW/dx$,

$$V_\pm(x) = \tfrac{1}{2}(W^2(x) \pm W'(x)). \tag{2.2}$$

Introducing the supercharge operators

$$A = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + W(x)\right), \qquad A^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + W(x)\right) \tag{2.3}$$

the SUSY partner Hamiltonians factorize as

$$H_+ = AA^\dagger \geqslant 0, \qquad H_- = A^\dagger A \geqslant 0 \tag{2.4}$$

and obviously obey the relation (1.1). As a consequence H_+ and H_- are essentially isospectral, that is, their strictly positive energy eigenvalues coincide. In addition one of the two Hamiltonians may have a vanishing eigenvalue. In this case, SUSY is said to be unbroken and by convention [3] (via an appropriate choice of an overall sign in W) this ground state then belongs to H_-. This convention implies that $\exp\{\int dx\, W(x)\} \notin \mathscr{H}$.

Let us be more explicit and denote the eigenfunctions and eigenvalues of $H_\pm$ by $\psi_n^\pm$ and $E_n^\pm$, respectively. That is,

$$H_\pm \psi_n^\pm(x) = E_n^\pm \psi_n^\pm(x), \quad n = 0, 1, 2, \ldots. \tag{2.5}$$

For simplicity we consider only the discrete part of the spectrum here. However, relations similar to those given below are also valid for the continuous part. In the case of unbroken SUSY (within the aforementioned convention) the zero-energy eigenstate of the SUSY system belongs to H_- and the corresponding ground state has the properties

$$E_0^- = 0, \quad \psi_0^-(x) = C \exp\left\{ -\int dx\, W(x) \right\} \in \mathscr{H} \tag{2.6}$$

with C denoting the normalization constant. The remaining spectrum of H_- coincides with the complete spectrum of H_+ and the corresponding eigenfunctions are related by SUSY transformations which are generated by the supercharge operators (2.3):

$$E_{n+1}^- = E_n^+ > 0, \quad \psi_{n+1}^-(x) = (E_n^+)^{-1/2} A^\dagger \psi_n^+(x), \quad \psi_n^+(x) = (E_{n+1}^-)^{-1/2} A \psi_{n+1}^-(x). \tag{2.7}$$

In the case of broken SUSY, H_+ and H_- are strictly isospectral and the eigenfunctions are also related by SUSY transformations:

$$E_n^- = E_n^+ > 0, \qquad \psi_n^-(x) = (E_n^+)^{-1/2} A^\dagger \psi_n^+(x), \qquad \psi_n^+(x) = (E_n^-)^{-1/2} A \psi_n^-(x). \tag{2.8}$$

With the help of the relations (2.6) and (2.7) or (2.8) it is obvious that knowing the spectral properties of, say H_+, one immediately obtains the complete spectral properties of the SUSY partner Hamiltonian H_-. These facts will be our basis for the general construction method of conditionally exactly solvable potentials, by which we mean that the eigenvalues and eigenfunctions of the corresponding Schrödinger Hamiltonian can be given in an explicit closed form (under certain conditions obeyed by the potential parameters).

3. THE CONSTRUCTION METHOD

In this section we present a rather general method for the construction of conditionally exactly solvable potentials using the SUSY transformations between the eigenstates of the SUSY partner Hamiltonians $H_\pm$. The basic idea is as follows. Let us look for some SUSY potential W such that under certain conditions on its parameters the corresponding partner potential V_+ becomes an exactly solvable one. For example, one of the shape-invariant potentials knowns form the factorization method [2, 3]. As a consequence the spectral properties of the associated Hamiltonian H_+ are known exactly. From the given SUSY potential W also follows the corresponding partner potential V_- and its associate Hamiltonian H_-. As we will see below, this potential is in general not shape-invariant but still exactly solvable via the SUSY transformations (2.7) or (2.8).

In order to find an appropriate class of SUSY potentials we make the ansatz

$$W(x) = \Phi(x) + f(x), \tag{3.1}$$

where Φ is chosen such that for $f \equiv 0$ the corresponding partner potentials $V_\pm$ belong to the known class of shape-invariant exactly solvable ones. For a non-vanishing f we have

$$V_+(x) = \tfrac{1}{2}[\Phi^2(x) + \Phi'(x) + f^2(x) + 2\Phi(x)\, f(x) + f'(x)] \tag{3.2}$$

If we now impose on f the condition that it obeys the following generalized Riccati equation

$$f^2(x) + 2\Phi(x)\, f(x) + f'(x) = b, \tag{3.3}$$

where, for the moment, b is assumed to be an arbitrary real constant, then the two partner potentials take the form

$$V_+(x) = \tfrac{1}{2}[\Phi^2(x) + \Phi'(x) + b], \tag{3.4}$$

$$V_-(x) = \tfrac{1}{2}[\Phi^2(x) - \Phi'(x) - 2f'(x) + b]. \tag{3.5}$$

Obviously, V_+ is, up to the additive constant $b/2$, a shape-invariant potential and therefore exactly solvable. With the help of the SUSY transformation we can now also solve the eigenvalue problem for H_- for the above given potential V_- which, due to the additional x-dependent term f' will in general be a new non-shape-invariant potential. At this step we already realize that the free parameter b has to be bounded below, as SUSY requires a strictly positive Hamiltonian H_+. This is a first condition on a parameter contained in V_- and already justifies to call it a conditionally exactly solvable (CES) potential.

The crucial problem in finding new CES potentials is to find the most general solution of the generalized Riccati Eq. (3.3). For this reason we linearize this equation by making the ansatz

$$f(x) = \frac{d}{dx} \log u(x) = \frac{u'(x)}{u(x)}, \tag{3.6}$$

which brings it into the form of a homogeneous linear second-order differential equation

$$u''(x) + 2\Phi(x)\, u'(x) - bu(x) = 0. \tag{3.7}$$

The general solution of this equation is given by a linear combination of two linearly independent fundamental solutions

$$u(x)=\alpha u_1(x)+\beta u_2(x). \tag{3.8}$$

Hence, besides the parameters contained in Φ and the parameter b the new CES potential V_- will also depend on the real parameters α and β. Note, however, that only the quotient α/β or β/α will enter V_- as a relevant parameter. In other words, depending on the actual situation one of these two parameters can be chosen (without loss of generality) to unity. The remaining parameters, however, will in general be not arbitrary real numbers and have to be chosen such that the corresponding supercharges

$$A=\frac{1}{\sqrt{2}}\left(\frac{d}{dx}+\Phi(x)+\frac{u'(x)}{u(x)}\right), \qquad A^\dagger=\frac{1}{\sqrt{2}}\left(-\frac{d}{dx}+\Phi(x)+\frac{u'(x)}{u(x)}\right) \tag{3.9}$$

are well-defined operators leaving the Hilbert space invariant, $A\colon \mathscr{H}\to\mathscr{H}$, $A^\dagger\colon \mathscr{H}\to\mathscr{H}$. A sufficient condition for that is to allow only for nonvanishing solutions (3.8). Thus the parameters have to be chosen such that u is (without loss of generality) a strictly positive function. Indeed, this condition also guarantees us that the potential

$$V_-(x)=\frac{1}{2}\Phi^2(x)-\frac{1}{2}\Phi'(x)+\frac{u'(x)}{u(x)}\left(2\Phi(x)+\frac{u'(x)}{u(x)}\right)-\frac{b}{2} \tag{3.10}$$

does not have singularities inside the configuration space. So H_+ and H_- have indeed a common domain $\mathscr{H}$. This condition is actually the most difficult part in our approach.

For all shape-invariant SUSY potentials, which we have considered, Eq. (3.7) can be reduced to a hypergeometric or confluent hypergeometric differential equation. That is, the two fundamental solutions u_1 and u_2 in (3.8) are expressed in terms of hypergeometric or confluent hypergeometric functions. Finding the proper linear combination leading to a strictly positive solution is very difficult and in general can be obtained only by inspection (numerically and/or via the asymptotic behaviour at the boundaries of the configuration space).

Besides the above mentioned necessary conditions on the potential parameters b, α, β, and possible additional ones contained in Φ, we will further restrict these parameters in the following respect. Let us assume that the SUSY quantum system (2.1) is unbroken (broken) for $f=0$. Then we consider only those values of the parameters for which the system with $f\neq 0$ remains to have unbroken (broken) SUSY. Hence, due to our ground-state convention, we have the following additional conditions:

$$
\begin{aligned}
&\exp\left\{\int dx\, W(x)\right\} = u(x)\exp\left\{\int dx\, \Phi(x)\right\} \notin \mathcal{H} \\
&\qquad\qquad \text{for broken and unbroken SUSY,} \\
&\exp\left\{-\int dx\, W(x)\right\} = [u(x)]^{-1}\exp\left\{-\int dx\, \Phi(x)\right\} \notin \mathcal{H} \\
&\qquad\qquad \text{for broken SUSY,} \\
&\exp\left\{-\int dx\, W(x)\right\} = [u(x)]^{-1}\exp\left\{-\int dx\, \Phi(x)\right\} \in \mathcal{H} \\
&\qquad\qquad \text{for unbroken SUSY.}
\end{aligned}
\tag{3.11}
$$

In the following we will consider several examples on the real line, the positive half line, and a finite interval. Both, unbroken as well as broken SUSY systems will be discussed.

4. QUANTUM SYSTEMS ON THE REAL LINE

In this section we will consider two examples on the real line in some detail. These are the linear and the Morse oscillator, which both have an unbroken SUSY. Note that there are no known shape-invariant potentials on $\mathbb{R}$ which allow for a broken SUSY. Finally, we also briefly summarize some results for the symmetric Rosen–Morse oscillator.

4.1. *The Linear Harmonic Oscillator*

The first SUSY system we are considering is characterized by a linear SUSY potential $\Phi(x) = x$ which gives rise to a unbroken SUSY with potential

$$V_+(x) = \tfrac{1}{2}(x^2 + b + 1). \tag{4.1}$$

The energy eigenvalues and eigenfunctions of the corresponding Hamiltonian read

$$E_n^+ = n + b/2 + 1, \qquad \psi_n^+(x) = [\sqrt{\pi}\, 2^n n!]^{-1/2} H_n(x) \exp\{-x^2/2\}, \tag{4.2}$$

where H_n denotes the Hermite polynomial of order $n \in \mathbb{N}_0$. Clearly, positivity of H_+ implies the condition $b > -2$.

The general solution of (3.7) can be given in terms of confluent hypergeometric functions [20],

$$u(x) = \alpha\, {}_1F_1\left(-\frac{b}{4}, \frac{1}{2}, -x^2\right) + \beta x\, {}_1F_1\left(\frac{2-b}{4}, \frac{3}{2}, -x^2\right), \tag{4.3}$$

and has the following asymptotic behaviour for $x \to \pm\infty$

$$u(x) = |x|^{b/2} \left(\alpha \frac{\Gamma(1/2)}{\Gamma((b+2)/4)} + \beta \frac{\Gamma(3/2)}{\Gamma(b/4+1)} \right) (1 + O(|x|^{-1})). \tag{4.4}$$

Here and in the following Γ denotes Euler's gamma function. From this asymptotic behaviour the condition on the parameters α and β for a strictly nonvanishing u reads $|\beta/\alpha| < 2\Gamma(b/4+1)/\Gamma((b+2)/4)$. Note that the right-hand side of this inequality is positive as $b > -2$ and that α must not vanish, that is, it can be chosen equal to unity, $\alpha = 1$.

The potential V_- can be obtained from (3.10) and explicitly reads

$$V_-(x) = \frac{1}{2} x^2 - \frac{b+1}{2} + \frac{u'(x)}{u(x)} \left[2x + \frac{u'(x)}{u(x)} \right], \tag{4.5}$$

where u is given in (4.3). The eigenvalues and eigenfunction for the associated partner Hamiltonian H_- are found via (2.6) and (2.7) as SUSY remains unbroken:

$$\begin{aligned} E_0^- &= 0, \\ \psi_0^-(x) &= \frac{C}{u(x)} \exp\{-x^2/2\}, \\ E_{n+1}^- &= E_n^+, \\ \psi_{n+1}^-(x) &= \frac{\exp\{-x^2/2\}}{[\sqrt{\pi}\, 2^{n+1} n!\, (n+b/2+1)]^{1/2}} \left(H_{n+1}(x) + H_n(x) \frac{u'(x)}{u(x)} \right). \end{aligned} \tag{4.6}$$

Figure 1 presents a graph of this family of potentials for $b \in [-2.5, 3]$, $\alpha = 1$, and $\beta \equiv \beta(b) = 1.5 \times \Gamma(b/4+1)/\Gamma((b+2)/4)$. It clearly shows singularities for $b \leqslant -2$ as expected. In Fig. 2 we keep $b = -1.9$ fixed and display the potential V_- for various values of the asymmetry parameter β. Again singularities appear for $|\beta| \geqslant 2\Gamma(b/4+1)/\Gamma((b+2)/4) \simeq 0.08569$. Let us note here that the potential (4.5) has previously been considered by Hongler and Zheng [21] in connection with an exactly solvable Fokker–Planck problem, which is closely related to Witten's SUSY quantum mechanics [3].

Special cases of V_- have also previously been found with the methods mentioned in the Introduction. For example, the special values $b = 0$, $\alpha = \gamma$, and $\beta = 1$ lead to $u(x) = \gamma + (\sqrt{\pi}/2)\,\mathrm{Erf}(x)$ (Erf denotes the error function) which is the result of Mielnik [9]. For $b = 4N$, $N \in \mathbb{N}$, $\alpha = 1$, and $\beta = 0$ the conditionally exactly solvable potential reads

$$V_-(x) = \frac{x^2}{2} + 8N(2N-1) \frac{H_{2N-2}(ix)}{H_{2N}(ix)} - 16N^2 \left(\frac{H_{2N-1}(ix)}{H_{2N}(ix)} \right)^2 + 2N - \frac{1}{2} \tag{4.7}$$

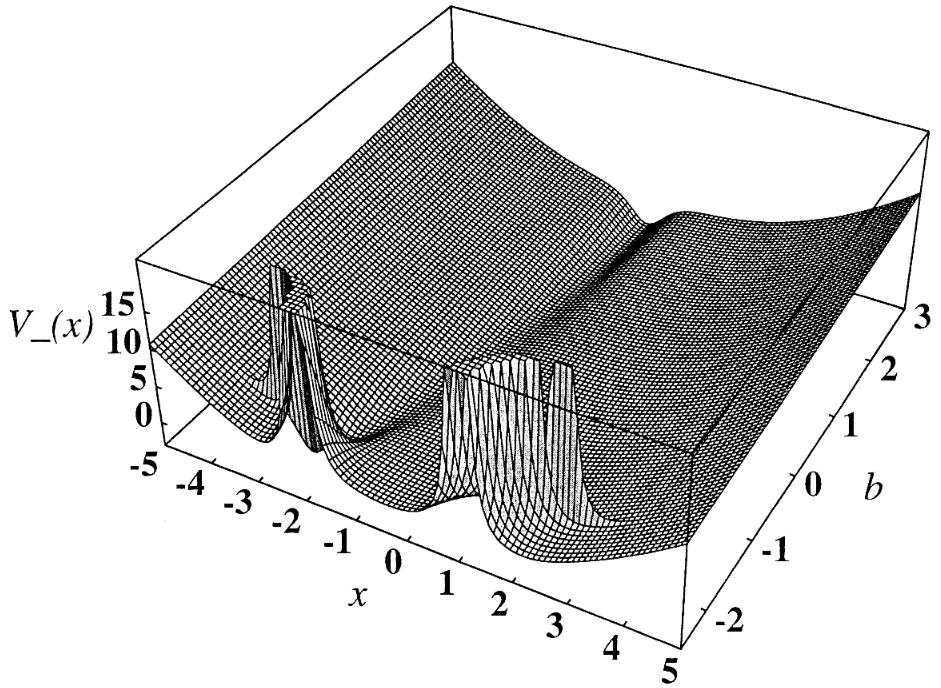

FIG. 1. The potential (4.5) for fixed $\alpha = 1$, $\beta = 1.5 \times \Gamma(b/4 + 1)/\Gamma((b+2)/4)$, and various ranges of the parameter b. Note that for $b \leqslant -2$ the potential exhibits singularities due to the existence of zeros in u as given in (4.3).

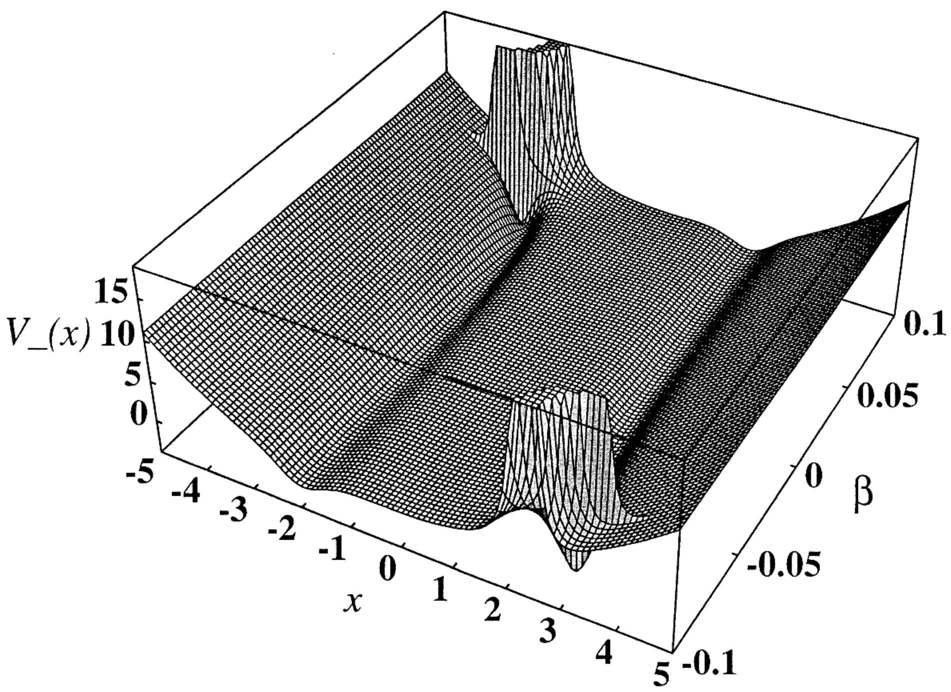

FIG. 2. The potential (4.5) for fixed $\alpha = 1$, $b = -1.9$, and various values of the asymmetry parameter β. Here values of β with $|\beta| \geqslant 0.08569$ violate the positivity condition for u (see text) and thus lead to singularities in V_-.

which has previously been obtained by Bagrov and Samsonov [16] via the N-order Darboux method. See also [18] where, in particular, the cases $N=1$ and 2 have been discussed.

4.2. *The Morse Oscillator*

As a second example we consider the Morse oscillator which is characterized by the SUSY potential

$$\Phi(x)=\gamma-e^{-x}, \qquad \gamma>0, \tag{4.8}$$

where the condition on the parameter γ results from our ground-state convention (see Section 2). Changing from parameter b to

$$\rho=\sqrt{\gamma^2+b} \tag{4.9}$$

the corresponding potential (3.4) reads

$$V_+(x)=\tfrac{1}{2}(e^{-2x}-(2\gamma-1)\,e^{-x}+\rho^2). \tag{4.10}$$

The (discrete) spectral properties of the associated Hamiltonian H_+ are

$$\begin{aligned} E_n^+ &= -\frac{1}{2}(\gamma-n-1)^2+\frac{\rho^2}{2}, \qquad n=0,1,2,\ldots<\gamma-1,\\ \psi_n^+(x) &= \left[\frac{(2\gamma-2n-2)\,\Gamma(n+1)}{\Gamma(2\gamma-n-1)}\right]^{1/2} 2^{\gamma-n-1}\\ &\quad\times\exp\{-e^{-x}-x(\gamma-n-1)\}\,L_n^{(2\gamma-2n-2)}(2e^{-x}), \end{aligned} \tag{4.11}$$

with L_n^ν denoting the generalized Laguerre polynomial of order n [20]. Obviously, positivity of H_+ implies the condition

$$\rho>\gamma-1. \tag{4.12}$$

With the above SUSY potential (4.8) the differential equation (3.7) can be reduced to that of the confluent hypergeometric equation and in turn the general solution reads

$$u(x)=\alpha e^{-x(\gamma+\rho)}{}_1F_1(\gamma+\rho,\,1+2\rho,\,-2e^{-x})+\beta e^{-x(\gamma-\rho)}{}_1F_1(\gamma-\rho,\,1-2\rho,\,-2e^{-x}), \tag{4.13}$$

which has the following asymptotic behaviour for $x\to-\infty$:

$$u(x)=\alpha\frac{\Gamma(1+2\rho)}{2^{\gamma+\rho}\Gamma(1-\gamma+\rho)}+\beta\frac{\Gamma(1-2\rho)}{2^{\gamma-\rho}\Gamma(1-\gamma-\rho)}+O(e^x). \tag{4.14}$$

From the asymptotic behaviour of u for $x \to +\infty$, which can trivially be extracted from (4.13), and the form of the SUSY ground-state wavefunction

$$\psi_0^-(x) = \frac{C}{u(x)} \exp\{-\gamma x - e^{-x}\} \tag{4.15}$$

one finds that SUSY remains unbroken iff $\beta \neq 0$. Hence, we can set it equal to unity, $\beta = 1$. The positivity condition of u can, with the help of the relation (4.14), be translated into conditions on the remaining parameters. These are

$$\rho > \gamma - 1, \qquad \frac{\Gamma(1-2\rho)}{\Gamma(1-\rho-\gamma)} > 0, \qquad \alpha > -2^{2\rho} \frac{\Gamma(1-2\rho)\,\Gamma(1+\rho-\gamma)}{\Gamma(1+2\rho)\,\Gamma(1-\rho-\gamma)}, \tag{4.16}$$

which have to be obeyed simultaneously.

In Fig. 3 we have shown the family of potentials

$$V_-(x) = \frac{1}{2} e^{-2x} - \left(\gamma + \frac{1}{2}\right) e^{-x} + \gamma^2 - \frac{\rho^2}{2} + \frac{u'(x)}{u(x)} \left(2\gamma - 2e^{-x} + \frac{u'(x)}{u(x)}\right) \tag{4.17}$$

for $\alpha = 0$, $\gamma = 1$, and $\rho \in [0, 4]$. Note that from (4.16) the allowed values of ρ for the given α and γ are $\rho \in \bigcup_{k=0}^{\infty}]2k + 1/2, 2k + 3/2[$. These admissible ranges of ρ are clearly visible in Fig. 3. Figure 4 shows the graph of V_- for the cases $\alpha \neq 0$

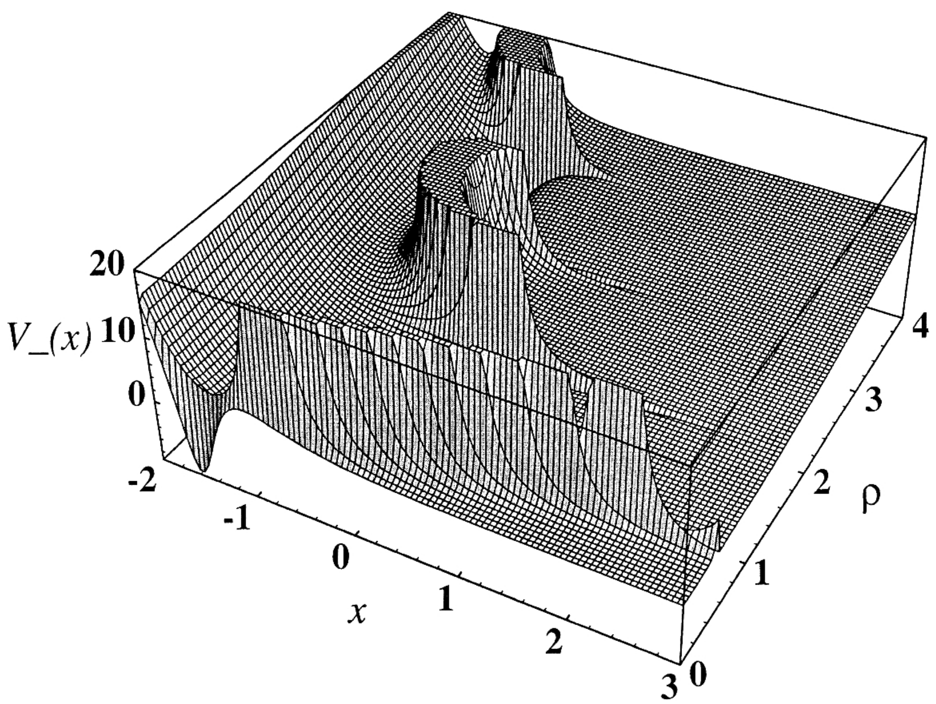

FIG. 3. The CES potentials of the Morse oscillator. Here V_- is shown for $\alpha = 0$, $\gamma = 1$, and $\rho \in [0, 4]$. The corresponding solution u is given in (4.13). Note the appearance of singularities in V_- due to the violation of the conditions given in (4.16).

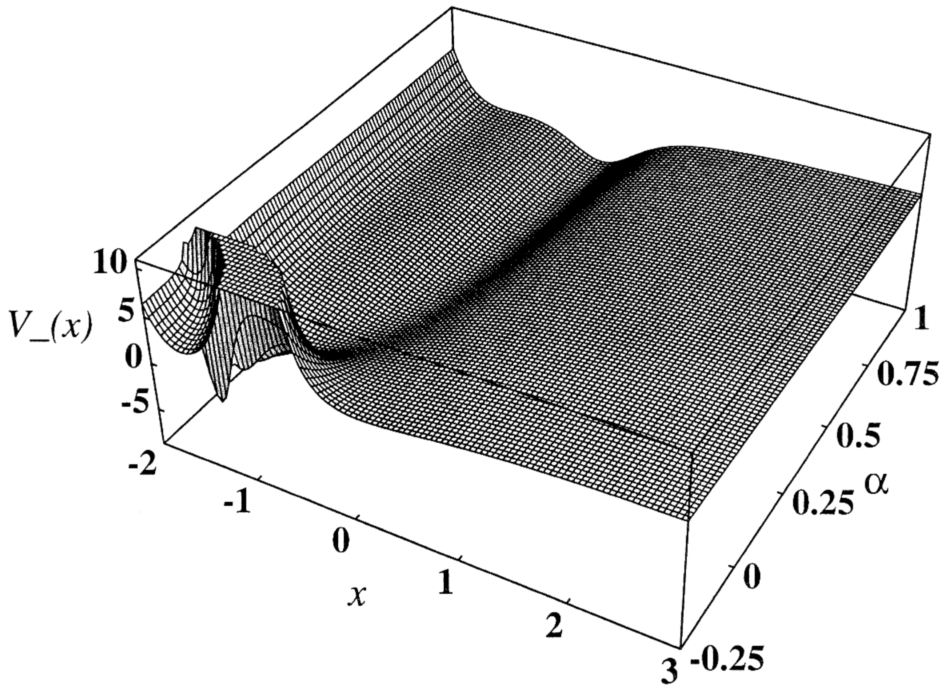

FIG. 4. Same as Fig. 4 but now for fixed $\gamma=\rho=3$ and various values of α. Again singularities appear for $\alpha \leqslant -4/45=-0.08889$ due to the last condition in (4.16).

and $\gamma=\rho=3$. Note that the last condition in (4.16) now explicitly reads $\alpha > -4/45=-0.08889$. The violation of this condition is also clearly visible in Fig. 4 via the singularities in V_-.

To complete the discussion of this example we finally give the discrete spectral properties of the corresponding partner Hamiltonian H_-. As SUSY remains unbroken the ground-state energy vanishes, $E_0^-=0$, and the corresponding eigenfunction is given in (4.15). For the excited states the discrete spectrum is given by $E_{n+1}^-=E_n^+$ and the associated wavefunctions explicitly read

$$\psi_{n+1}^-(x)=\left[\frac{(2\gamma-2n-2)\,\Gamma(n+1)}{(\rho^2-(\gamma-n-1)^2)\,\Gamma(2\gamma-n-1)}\right]^{1/2} 2^{\gamma-n-1}\exp\{-e^{-x}-x(\gamma-n-1)\}$$
$$\times\left[(n+1)\,L_{n+1}^{(2\gamma-2n-2)}(2e^{-x})+\frac{u'(x)}{u(x)}\,L_n^{(2\gamma-2n-2)}(2e^{-x})\right]. \qquad (4.18)$$

4.3. *The Symmetric Rosen–Morse Oscillator*

As a last example on the real line let us briefly discuss the symmetric Rosen–Morse potential (sometimes also called modified Pöschl–Teller potential) which is characterized by the SUSY potential

$$\Phi(x)=\gamma\tanh(x), \qquad \gamma>0. \qquad (4.19)$$

The corresponding potential V_+ reads

$$V_+(x) = -\frac{\gamma(\gamma-1)}{2\cosh^2 x} + \frac{\gamma^2+b}{2} \tag{4.20}$$

and for $\gamma \in \mathbb{N}$ is known to belong to the class of reflectionless potentials, which are, for example, important for the construction of explicit solutions of the Korteweg–deVries equation [22].

For the above SUSY potential (3.7) can be reduced to Legendre's differential equation and the general solution is given by

$$u(x) = \cosh^{-\gamma}(x)[\alpha P_{\gamma-1}^{(\gamma^2+b)^{1/2}}(\tanh x) + \beta Q_{\gamma-1}^{(\gamma^2+b)^{1/2}}(\tanh x)], \tag{4.21}$$

where P_ν^μ and Q_ν^μ denote Legendre functions as defined in [20]. We leave it to the reader to investigate the proper admissible ranges for the potential parameters b, α, β, and γ, and only remark that the family of partner potentials

$$V_-(x) = -\frac{\gamma(\gamma+1)}{2\cosh^2 x} + \frac{\gamma^2-b}{2} + \frac{u'(x)}{u(x)}\left(2\gamma \tanh x + \frac{u'(x)}{u(x)}\right) \tag{4.22}$$

will contain new reflectionless potentials (via the choice $\gamma \in \mathbb{N}$) and thus may, for example, allow us to find new explicit solutions for the Korteweg–deVries equation.

5. QUANTUM SYSTEMS ON THE POSITIVE HALF LINE

As examples of new CES potentials on the positive half line we consider in this section the radial harmonic oscillator, which allows for unbroken as well as broken SUSY, and the radial hydrogen atom problem.

5.1. *The Radial Harmonic Oscillator with Broken SUSY*

The SUSY potential for the radial harmonic oscillator is given by

$$\Phi(x) = x + \frac{\gamma}{x}. \tag{5.1}$$

This SUSY potential leads to an unbroken SUSY system ($f = 0$) if the parameter γ is negative. This case, which has already been discussed in some detail in [18], leads to rather strict conditions on the potential parameter b and in turn gives rise to a very limited class of new CES potentials. Therefore, we discuss here only the case of broken SUSY, that is, $\gamma > 0$.

The potential for the partner Hamiltonian H_+ reads

$$V_+(x) = \frac{x^2}{2} + \frac{\gamma(\gamma-1)}{2x^2} + \gamma + \frac{b+1}{2} \tag{5.2}$$

and gives rise to the following spectral properties

$$E_n^+ = 2n + 2\gamma + 1 + \frac{b}{2}, \qquad \psi_n^+(x) = \left[\frac{2n!}{\Gamma(n+\gamma+1/2)}\right]^{1/2} x^\gamma e^{-x^2/2} L_n^{(\gamma-1/2)}(x^2). \quad (5.3)$$

Positivity of H_+ leads us to the condition $b > -4\gamma - 2$.

The general solution of Eq. (3.7) is expressed in terms of the confluent hypergeometric function and reads

$$u(x) = \alpha\, {}_1F_1\left(-\frac{b}{4}, \gamma + \frac{1}{2}, -x^2\right) + \beta x^{1-2\gamma}\, {}_1F_1\left(\frac{1}{2} - \frac{b}{4} - \gamma, \frac{3}{2} - \gamma, -x^2\right). \quad (5.4)$$

For small $0 < x \ll 1$ this solution behaves like $u(x) \approx (\alpha + \beta x^{1-2\gamma})(1 + O(x^2))$ and as a consequence we have to set $\beta = 0$ for SUSY to remain broken. Note that $\exp\{-\int dx\, W(x)\} = \exp\{-x^2/2\}/x^\gamma u(x)$ and cf. Eq. (3.11). Therefore, without loss of generality we set $\alpha = 1$ and consider from now on only the solution

$$u(x) = {}_1F_1\left(-\frac{b}{4}, \gamma + \frac{1}{2}, -x^2\right) = e^{-x^2}\, {}_1F_1\left(\gamma + \frac{b+2}{4}, \gamma + \frac{1}{2}, x^2\right) \quad (5.5)$$

leading to broken SUSY. This solution will have no zeros if $b > -4\gamma - 2$, a condition which we have found before from the positivity of H_+.

The partner potential reads

$$V_-(x) = \frac{x^2}{2} + \frac{\gamma(\gamma+1)}{2x^2} + \gamma - \frac{b+1}{2} + \frac{u'(x)}{u(x)}\left(2x + \frac{2\gamma}{x} + \frac{u'(x)}{u(x)}\right) \quad (5.6)$$

and is shown in Fig. 5 for $\gamma = 0.5$ and various values of b. As expected there are singularities in V_- for those values of b which violated the above condition. The eigenvalues of the corresponding Hamiltonian H_- are identical to those of H_+ given in (5.3) with eigenfunctions

$$\psi_n^-(x) = \left[\frac{2n!}{(n+\gamma+1/2+b/4)\,\Gamma(n+\gamma+1/2)}\right]^{1/2} x^{\gamma+1} e^{-x^2/2} \left(L_n^{(\gamma+1/2)}(x^2) + \frac{u'(x)}{2xu(x)}\right) \quad (5.7)$$

which follow from the SUSY transformation (2.8).

Finally, we note that for unbroken SUSY ($l = -\gamma > 0$) the special case $b = 0$ of (5.4)

$$u(x) = \alpha + 2\beta x^{2l+1} \int_0^x dt\, t^{2l} e^{-t^2} \quad (5.8)$$

has, in essence, been discussed before in [23, 24].

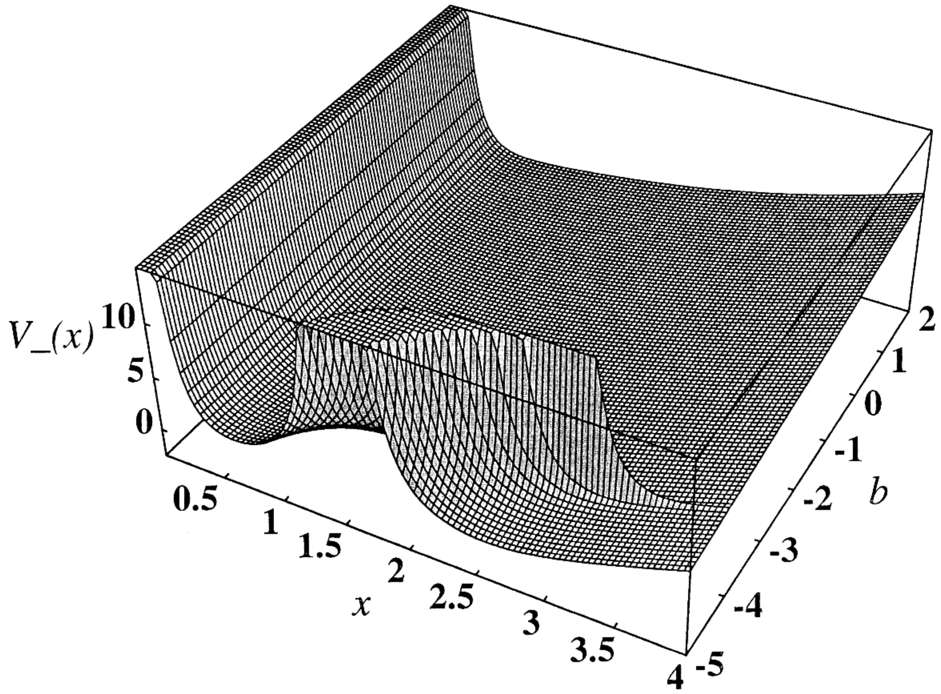

FIG. 5. The CES potential (5.6) of the radial harmonic oscillator for $\gamma = 0.5$ and various values of b. Note that the allowed range for this parameter is given by $b > -4\gamma - 2 = -4$.

5.2. *The Hydrogen Atom*

The SUSY potential for the radial hydrogen atom problem is given by

$$\Phi(x) = \frac{a}{\gamma} - \frac{\gamma}{x}, \qquad a, \gamma > 0, \tag{5.9}$$

and leads to the partner potential

$$V_+(x) = -\frac{a}{x} + \frac{\gamma(\gamma+1)}{2x^2} + \frac{1}{2}(b + a^2/\gamma^2). \tag{5.10}$$

The spectral properties of the associated partner Hamiltonian H_+ are well known. For simplicity we give here only the discrete eigenvalues

$$E_n^+ = -\frac{a^2}{2(n+\gamma+1)^2} + \frac{1}{2}(b + a^2/\gamma^2), \qquad n \in \mathbb{N}_0. \tag{5.11}$$

Then the positivity of H_+ leads to the condition

$$\rho = \sqrt{b + a^2/\gamma^2} > a/(\gamma+1). \tag{5.12}$$

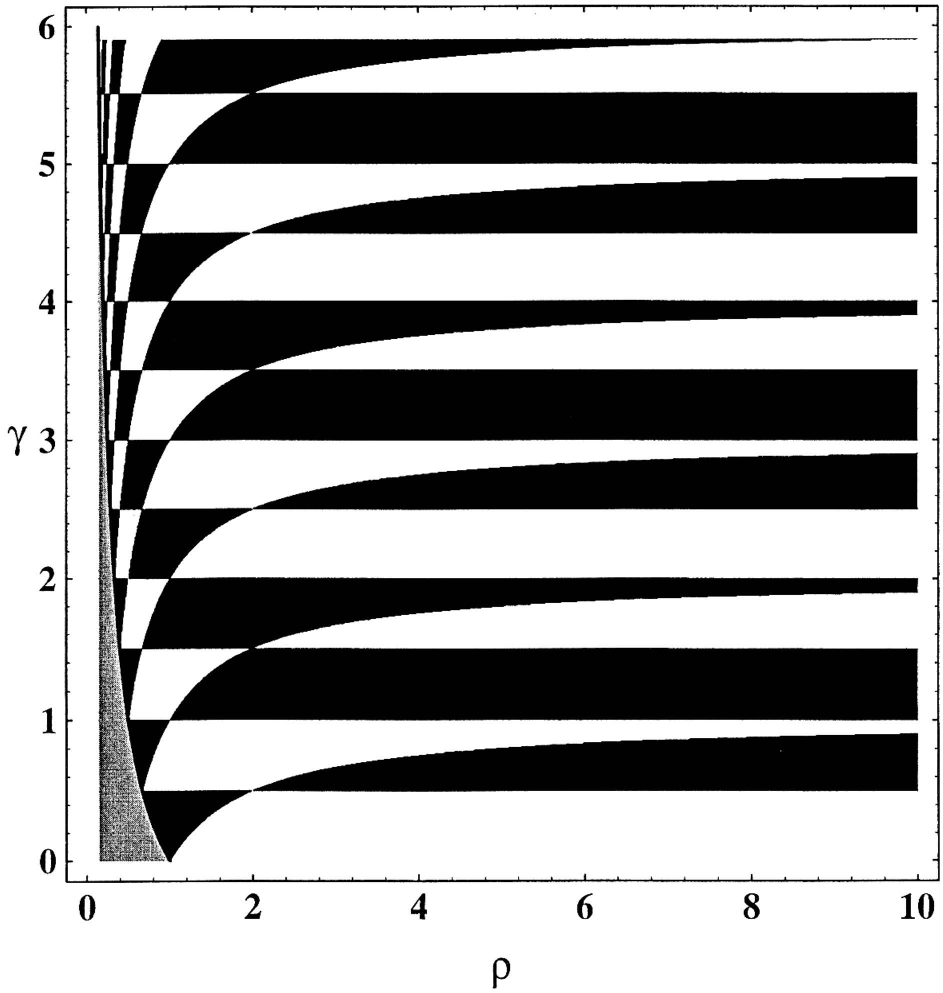

FIG. 6. Allowed ranges for the parameters γ and ρ of the hydrogen atom problem according to the first two conditions given in (5.16). For details see the text.

In the present case the general solution of (3.6) is again given in terms of confluent hypergeometric functions

$$u(x) = e^{-(a/\gamma+\rho)x}[\alpha \, {}_1F_1(-\gamma - a/\rho, \, -2\gamma, \, 2\rho x) \\ + \beta(2\rho x)^{2\gamma+1} \, {}_1F_1(\gamma+1-a/\rho, \, 2\gamma+2, \, 2\rho x)] \tag{5.13}$$

and has the following asymptotic form for large x

$$u(x) = (2\rho x)^{\gamma - a/\rho} \, e^{(\rho - a/\gamma)x} \left[\alpha \frac{\Gamma(-2\gamma)}{\Gamma(-\gamma - a/\rho)} + \beta \frac{\Gamma(2\gamma+2)}{\Gamma(\gamma+1-a/\rho)} \right] (1 + O(x^{-1})). \tag{5.14}$$

In order to find all conditions on the potential parameters we first note that

$$\psi_0^-(x) = \frac{C}{u(x)} \, x^\gamma e^{-ax/\gamma} \tag{5.15}$$

and, therefore, the parameter α must not vanish in order for SUSY to remain unbroken. Hence, without loss of generality we may put $\alpha = 1$. From the above asymptotic form (5.14) we can now also deduce further conditions on the parameters from the positivity restriction on u. Summarizing all conditions we have

$$\rho > \frac{a}{\gamma+1}, \qquad \frac{\Gamma(-2\gamma)}{\Gamma(-\gamma-a/\rho)} > 0, \qquad \beta > -\frac{\Gamma(-2\gamma)}{\Gamma(-\gamma-a/\rho)} \frac{\Gamma(\gamma+1-a/\rho)}{\Gamma(2\gamma+2)}. \tag{5.16}$$

In Fig. 6 we give a graphical representation of the first and second condition. Here the grey area shows the forbidden region due to the first condition and the black area the forbidden region due to the second condition. In other words, the allowed region of the two parameters γ and ρ for a given coupling constant a, which is set equal to unity in Fig. 6, is the white area.

The CES potential for the hydrogen atom problem reads

$$V_-(x) = -\frac{a}{x} + \frac{\gamma(\gamma-1)}{2x^2} + \frac{a^2}{\gamma^2} - \frac{\rho^2}{2} + \frac{u'(x)}{u(x)}\left(\frac{2a}{\gamma} - \frac{2\gamma}{x} + \frac{u'(x)}{u(x)}\right). \tag{5.17}$$

Figure 7 shows this potential for $a=1$, $\beta=0$, and $\gamma=2.8$. According to (5.16) the allowed region for ρ with the others as fixed above is given by $]\frac{5}{16}, \frac{5}{11}[\cup]\frac{5}{6}, 5[$. The singularity appearing for $\rho \geqslant 5$ is clearly visible in Fig. 7. The other singularities are outside the plotted range of $0 < x < 2$ and therefore not visible. In Fig. 8 we keep $\gamma = 2.8$, $a=1$, and $\rho = a/\gamma$ fixed and show the potential (5.17) for various values of β. Note the singularity appearing for $\beta \leqslant -4.39554 \times 10^{-4}$ according to

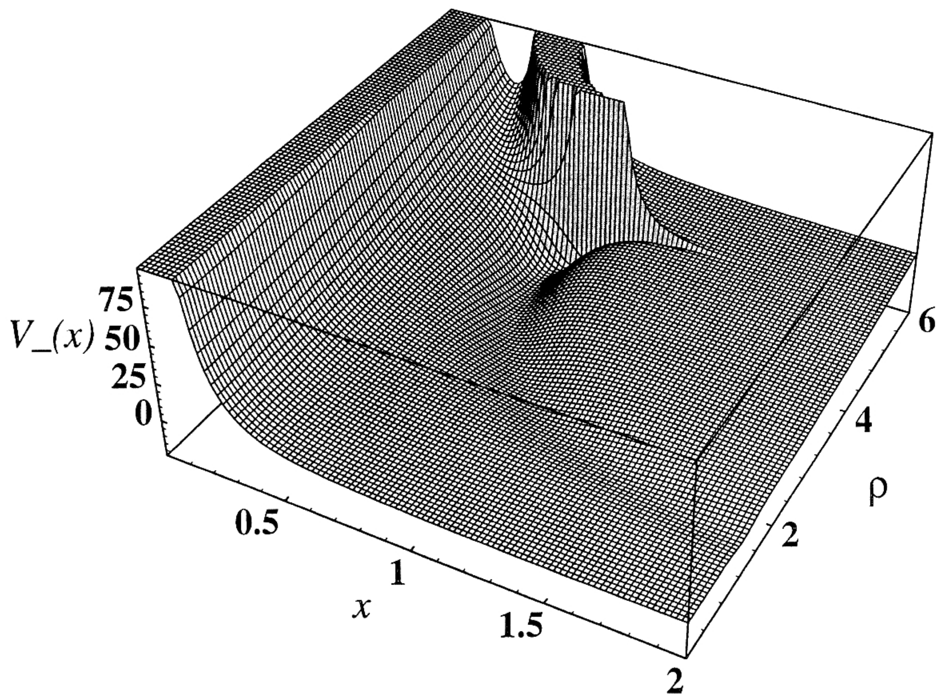

FIG. 7. The CES potential (5.17) of the hydrogen atom problem for $a=1$, $\beta=0$, $\gamma=2.8$, and various values of ρ.

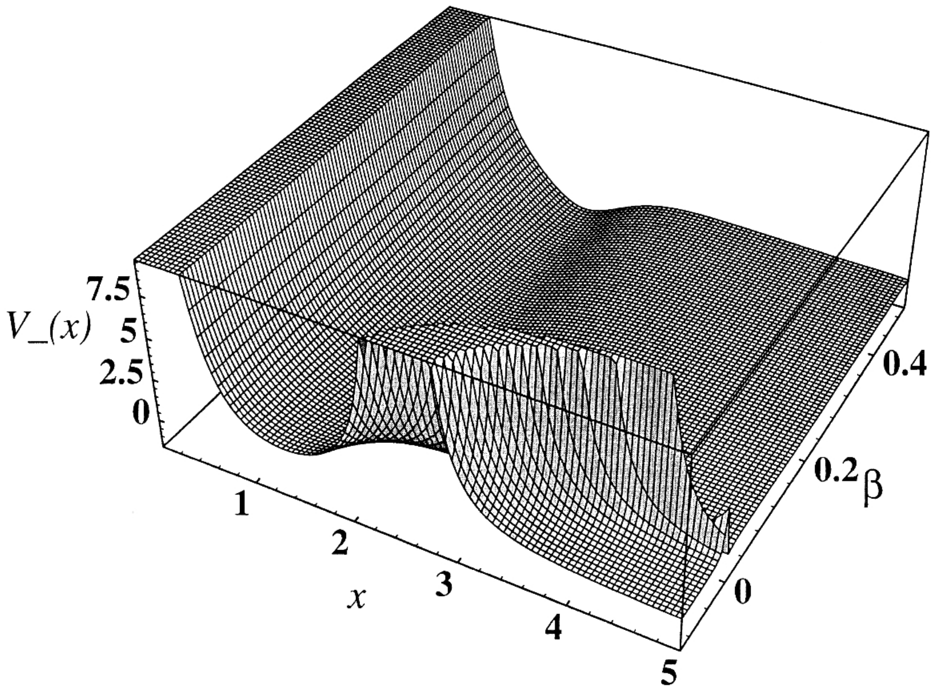

FIG. 8. Same as Fig. 7, but for fixed $a=1$, $\gamma=2.8$, $\rho=a/\gamma$, and various values of β.

the violation of the last condition in (5.16). Finally, let us also remark that for the special case $\rho=a/\gamma$ (i.e., $b=0$) we have

$$u(x)=\alpha+\beta(2\gamma+1)\int_0^{2ax/\gamma} dt\, t^{2\gamma}e^{-t} \tag{5.18}$$

a result, which has previously been found in [24].

6. QUANTUM SYSTEMS ON A FINITE INTERVAL

As example for a quantum system defined on a finite interval we will consider here the symmetric Pöschl–Teller potential, whose SUSY potential is given by

$$\Phi(x)=\gamma\tan x, \qquad \gamma>0, \tag{6.1}$$

leading to an unbroken SUSY with

$$V_+(x)=\frac{\gamma(\gamma+1)}{2\cos^2 x}+\frac{b-\gamma^2}{2}. \tag{6.2}$$

This is the well-studied Pöschl–Teller potential, which gives rise to the following spectral properties of H_+:

$$E_n^+ = \frac{1}{2}(\gamma+1+n)^2 + \frac{b-\gamma^2}{2}, \qquad n \in \mathbb{N}_0,$$
$$\psi_n^+(x) = \sqrt{\frac{(\gamma+1+n)\,\Gamma(2\gamma+2+n)}{\Gamma(n+1)}} \cos^{1/2} x P^{-\gamma-1/2}_{\gamma+n+1/2}(\sin x). \tag{6.3}$$

Again, positivity leads to a condition on the parameter b, $b > -2\gamma - 1$. However, for later convenience we introduce another parameter $\rho = \sqrt{\gamma^2 - b}$ and in terms of this, the above condition reads

$$0 \leqslant \rho < \gamma + 1 \qquad \text{or} \qquad \rho \in i\mathbb{R}. \tag{6.4}$$

The general solution for the corresponding differential Eq. (3.7) is given in terms of hypergeometric functions

$$u(x) = \alpha\, {}_2F_1\left(-\frac{\gamma+\rho}{2}, \frac{\gamma-\rho}{2}, \frac{1}{2}, \sin^2 x\right) + \beta \sin x\, {}_2F_1\left(\frac{1-\gamma-\rho}{2}, \frac{1-\gamma+\rho}{2}, \frac{3}{2}, \sin^2 x\right). \tag{6.5}$$

Obviously, as a necessary condition α must not vanish in order to have no zeros in this solution. Hence, we will set α equal to unity in the following discussion. From the values of u at the boundaries of the configuration space,

$$u(\pm\pi/2) = \frac{\Gamma(1/2)\,\Gamma(1/2+\gamma)}{\Gamma((1+\gamma+\rho)/2)\,\Gamma((1+\gamma-\rho)/2)} \times \left[1 \pm \frac{\beta}{2} \frac{\Gamma((1+\gamma+\rho)/2)\,\Gamma((1+\gamma-\rho)/2)}{\Gamma(1+(\gamma+\rho)/2)\,\Gamma(1+(\gamma-\rho)/2)}\right], \tag{6.6}$$

we also deduce a condition for the remaining parameter β:

$$|\beta| < 2\frac{\Gamma(1+(\gamma+\rho)/2)\,\Gamma(1+(\gamma-\rho)/2)}{\Gamma((1+\gamma+\rho)/2)\,\Gamma((1+\gamma-\rho)/2)}. \tag{6.7}$$

Finally, let us note that SUSY remains unbroken and the ground-state wave function for H_- is given by

$$\psi_0^-(x) = C\frac{\cos^\gamma x}{u(x)}. \tag{6.8}$$

Hence, (6.4) and (6.7) constitutes the complete set of conditions on the three parameters β, γ, and ρ. The corresponding partner potential is given by

$$V_-(x) = \frac{\gamma(\gamma-1)}{2\cos^2 x} - \gamma^2 + \frac{\rho^2}{2} + \frac{u'(x)}{u(x)}\left(2\gamma \tan x + \frac{u'(x)}{u(x)}\right), \tag{6.9}$$

which is shown in Figs. 9–11 for some special cases. In Figs. 9 and 10 we have set $\beta = 0$, $\gamma = 2$, and chosen real $(0 \leqslant \rho \leqslant 3.25)$ and purely imaginary $(0 \leqslant \rho/i \leqslant 4)$

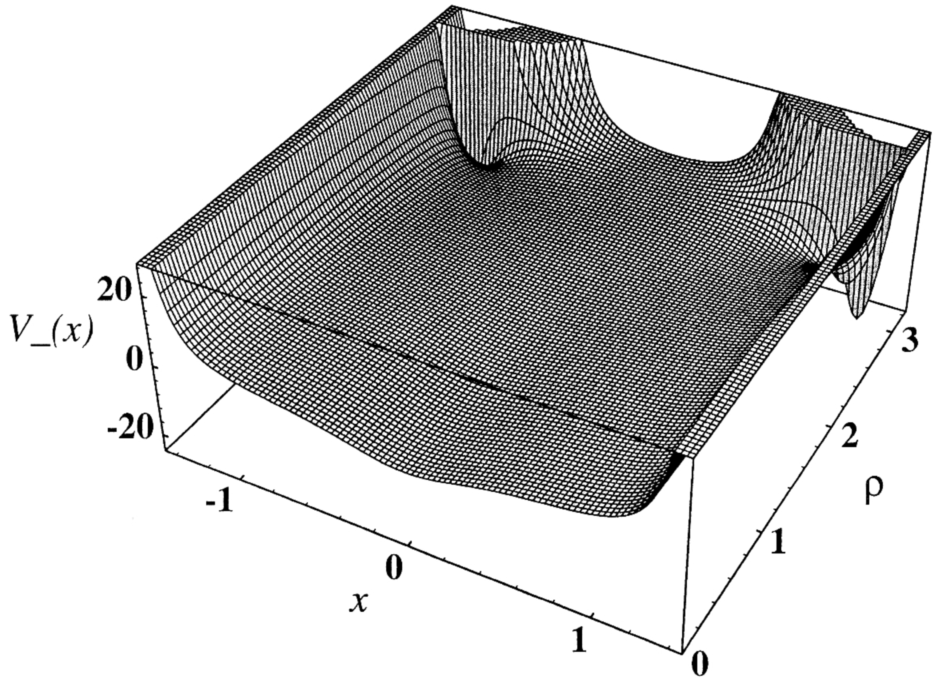

FIG. 9. The CES potential (6.9) of the Pöschl–Teller problem for $\beta = 0$, $\gamma = 2$, and $0 \leqslant \rho \leqslant 3.25$.

values for ρ, respectively. Figure 9 exhibits singularities for $\rho \geqslant 3$ as expected from (6.4), whereas Fig. 10 does not have singularities for the same reason. Finally, Fig. 11 shows the potential (6.9) for fixed $\gamma = 2$, $\rho = 1$, and various values of the asymmetry parameter β. Here due to condition (6.7) we expect and actually see singularities for $|\beta| \geqslant 2.35619$.

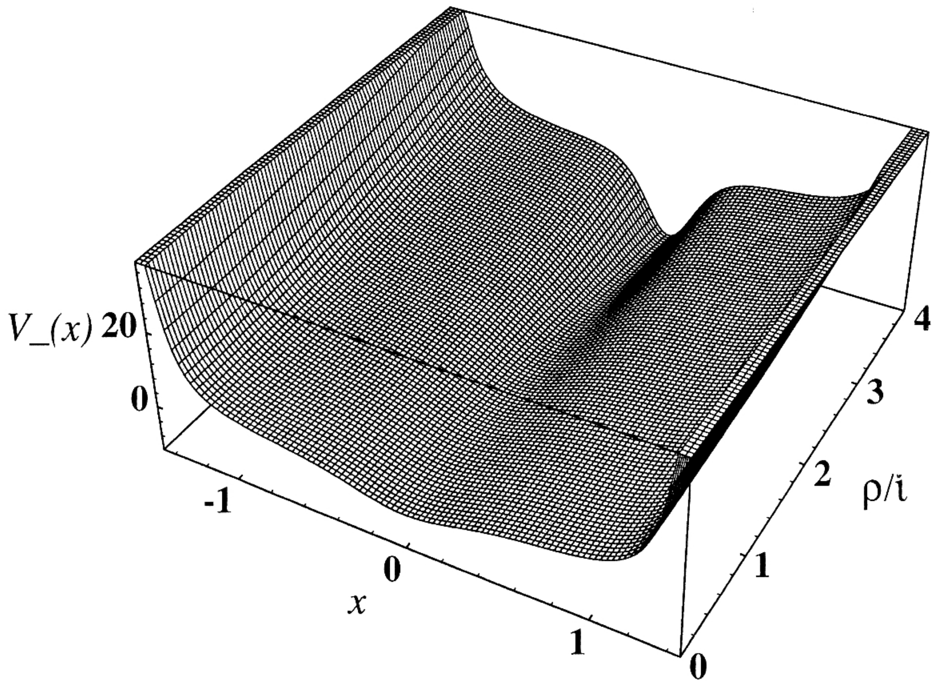

FIG. 10. Same as Fig. 9 but for complex ρ, $0 \leqslant \rho/i \leqslant 4$.

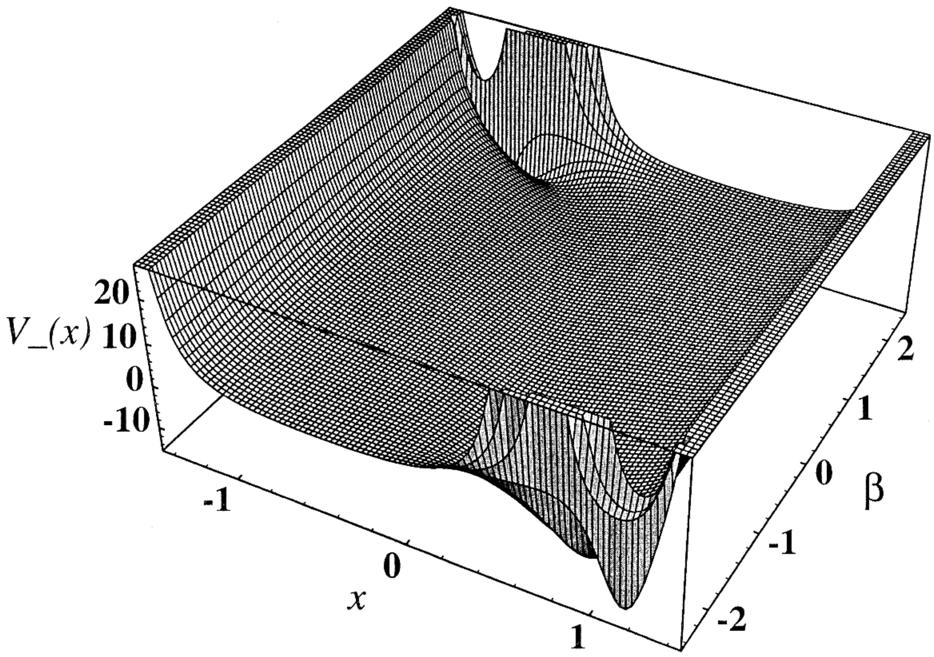

FIG. 11. Same as Fig. 9 with $\gamma = 2$ and $\rho = 1$ and various values of β.

7. CONCLUDING REMARKS

In this paper we have presented a method for constructing conditionally exactly solvable potentials starting from the known SUSY potentials of shape-invariant (exactly solvable) potentials. This method is more general then those given in the literature before. In particular, most of the previously constructed CES potentials correspond to the special case $b = 0$ of our method. We also remark that the new potentials constructed in the present work do not belong to the Natanzon class [5]. This is most obvious by noting the fact that in the present case of CES potentials the corresponding wave functions in general depend on the quotient u'/u which is a quotient of (confluent) hypergeometric functions. In contrast to this, the solutions of the Natazon class [5, 6] and their SUSY generalizations [7] are given by sums of (confluent) hypergeometric functions.

There are several ways to generalize the present approach. Obviously, one can now choose the newly found SUSY potentials of this paper as input and try to construct further CES potentials from these. In general we expect to find a hierarchy of new families of CES potentials belonging to the initial shape-invariant one. In the present paper we have restricted ourselves to those parameter values which conserve the nature of SUSY, that is, SUSY remains unbroken or broken by adding the $f = u'/u$ term to the SUSY potential. This condition can certainly be relaxed. Some of the conditions on the potential parameters have been extracted from the asymptotic behaviour of the solution u of (3.7). Hence, these conditions are only sufficient ones. In most cases we expect them to be also necessary, but there may be exceptions. In any case, if one wants to construct some exactly solvable model potential via the present method a detailed analysis of the allowed parameter values is advisable.

We should also note that the present approach can be utilized to construct new drift potentials for which the associated Fokker–Planck equation allows for an explicit and exact solution. This would be similar to the discussion of the linear harmonic oscillator by Hongler and Zheng [21]. Let us also mention that one may choose complex values for the parameters α and/or β. This will lead to complex partner potentials V_- whose associated non-hermitian Schrödinger Hamiltonian will have a real spectrum [25–27]. Finally, we note that all the known shape-invariant potentials give rise to a dynamical group structure [28]. This group structure induces, via the SUSY transformations (2.7)–(2.8), a related structure for the corresponding CES Hamiltonian H_-. For example, one can construct from the well-known ladder operators of the linear and radial harmonic oscillator via the supercharges (2.3) ladder operators for the corresponding partner Hamiltonian H_-. It turns out that these operators close a non-linear algebra [18]. A detailed discussion, in particular, of the coherent states associated with these non-linear algebras will be given elsewhere [29].

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